Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(01)00065-X
J. Appl. Maths Mechs, Vol. 65, No. 4, pp. 605–616, 2001

0021-8928/01/$—see front matter

THE STABILITY OF A CONFIGURATION OF A LIQUID LAYER IN AN AXIALLY SYMMETRIC CONTAINER UNIFORMLY ROTATING UNDER CONDITIONS OF WEIGHTLESSNESS†

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(*Received* 1 *August* 2000)

The stability of a layer of ideal incompressible liquid located on the wall of an axially symmetric container which uniformly rotates around its axis under conditions of weightlessness is investigated in the case when the free surface of the liquid is a circular cylinder. The stability of the relative equilibrium of the liquid mass depends on the coercivity of a certain bilinear form in the corresponding Hilbert space. The investigation of this coercivity is reduced to an auxiliary eigenvalue problem which is solved using the methods of functional and classical analysis. Two cases are considered depending on the curvature of the meridian of the container at the points where it comes into contact with the free surface of the liquid in relative equilibrium. The first of these cases is reduced to a classical eigenvalue problem and the second to a Steklov problem. Sufficient conditions for the stability of the relative equilibrium of the liquid, which depend on the curvature of the above mentioned meridian and on the angular velocity of the rotating container, are found. © 2001 Elsevier Science Ltd. All rights reserved.

The stability of the equilibrium and of the steady motion of a liquid mass placed in a moving or fixed container has been the subject of numerous papers which are described in the monographs [1, 2].‡ In particular, problems on the stability of the relative equilibria of a liquid mass in a reservoir which is rotating under weightlessness conditions have been considered in [2].

1. FORMULATION OF THE PROBLEM AND THE EQUATIONS OF MOTION OF THE LIQUID

Suppose an ideal incompressible liquid of density ρ is located in an axially symmetric container with an axis of symmetry z and a plane of symmetry xy, which rotates around the z axis with a constant angular velocity ω_0. The following notation is used: (r, θ, z) are cylindrical coordinates in which $r = f(z)$ is the equation of the meridian of the container (Fig. 1). We assume that the liquid is located on the walls of the container.

When the condition for the existence of a relative equilibrium when there are no gravitational forces is satisfied. The liquid pressure at a point $M(r, \theta, z)$ is defined as

$$P_0(M) = \rho\omega_0^2 r^2/2 + c \quad (c = \text{const})$$

Laplace's law has the form

$$P_0|_\Gamma - p_0 = -\alpha(k_1 + k_2)$$

†*Prikl. Mat. Mekh.* Vol. 65, No. 4, pp. 619–630, 2001.

‡Editor's note. The problem of the stability of the steady motion of a body with a capillary liquid has been reduced (see RUMYANTSEV, V. V., The stability of the motion of a rigid body with a liquid possessing surface tension, *Prikl. Mat. Mekh.* 1964, **28**, 4, 746–753) to the problem of the minimum of the augmented potential energy. This technique was subsequently used to investigate a number of specific problems (see, for example, SAMSONOV, V. A., Stability and bifurcation of the equilibrium of a body with a liquid. Nauchnye Trudy Inst. Mekh. Moskovsk. Gos. Univ. (MGU), Izd. Moskovsk. Univ., Moscow, No. 16, 1971; SLOBOZHANIN, L. A., The stability of the cylindrical state of a rotating fluid. In *Mathematical Physics and Functional Analysis*. Fivziko-Tekhn. Inst. Akad. Nauk UkrSSR, Khar'kov, Issue 2, 175–181).

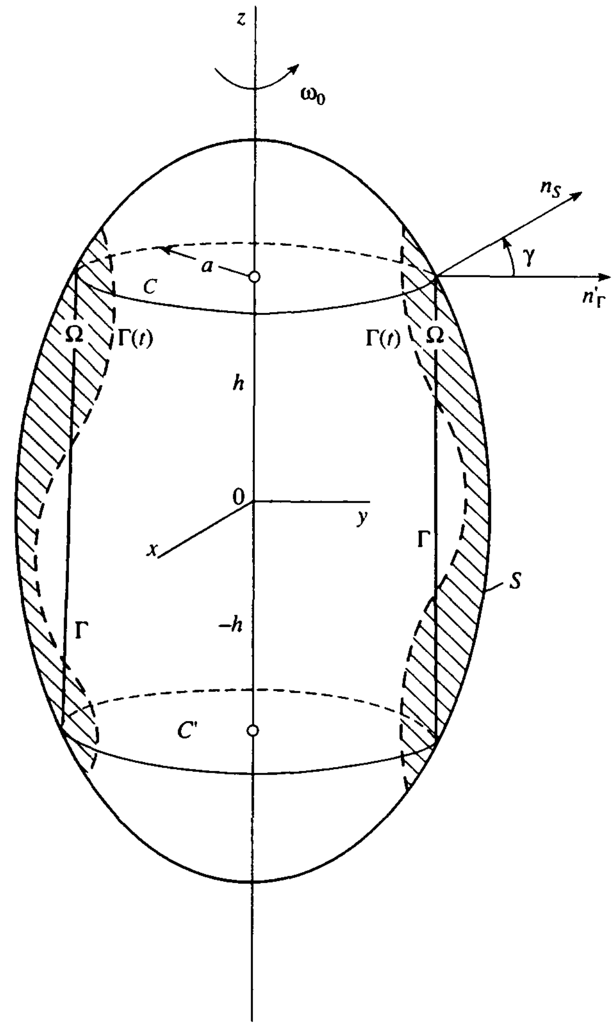

Fig. 1

where p_0 is the external pressure, α is the surface tension, which is assumed to be constant, and k_1^{-1} and k_2^{-1} are the radii of the principal curvatures of the free surface Γ, measured in a positive direction along the vector $\mathbf{n}_\Gamma$ of the unit normal, which is outward with respect to the fluid. The surface Γ is therefore defined by the partial differential equation

$$\rho\omega_0^2 r^2 / 2 + c - p_0 = -\alpha(k_1 + k_2)$$

The case is considered when Γ is part of a cylindrical surface $r = a$. In this case, $k_1 = 0$ and $k_2 = a^{-1}$. We then have

$$P_0(M) = \rho\omega_0^2 / 2(r^2 - a^2) + p_0 - \alpha / a$$

The domain occupied by the liquid in relative equilibrium is denoted by Ω, S is the wetted surface of the wall of the container, C and C' are the circles of intersection of the cylinder $r = a$ with the wall of the container, and h and $-h$ are the distances of the centres of the circles C and C' from the middle plane.

We will assume, in accordance with the capillarity law, that the free surface of the liquid intersects the container wall at a constant angle and we will denote the angle between the vectors $\mathbf{n}'_\Gamma$ and $\mathbf{n}_s$ by γ, where $\mathbf{n}_s$ is the vector, directed outside Ω, of the outward normal to the surface S with respect to the domain, and $\mathbf{n}'_\Gamma = -\mathbf{n}_\Gamma$. We have

$$a = f(h); \quad f_z(\pm h) = \mp \operatorname{tg} \gamma \quad (f_z = df/dz) \tag{1.1}$$

We will now write the equations for small motions of the liquid in the neighbourhood of the relative equilibrium position which is assumed to be stable.

Suppose $P(t, M)$ is the pressure at a point M at the instant of time t. We introduce the dynamic pressure

$$p(t, M) = P(t, M) - P_0(M)$$

Suppose $\mathbf{w}$ is the relative displacement of a liquid particle. When there are no bulk forces, Euler's equation and the incompressibility equation can be written in the form

$$\frac{\partial^2 \mathbf{w}}{\partial t^2} + 2\omega_0 \mathbf{z} \times \frac{\partial \mathbf{w}}{\partial t} + \frac{1}{\rho} \operatorname{grad} p = 0, \quad \operatorname{div} \mathbf{w} = 0 \tag{1.2}$$

To these equations, we add the impermeability condition of the liquid on the container wall

$$w_n |_S = \mathbf{w} \cdot \mathbf{n}_S = 0 \tag{1.3}$$

Laplace's law on the perturbed free surface $\Gamma(t)$ is written in the form

$$P|_{\Gamma(t)} - p_0 = -\alpha(\tilde{k}_1 + \tilde{k}_2)$$

where $\tilde{k}_1$ and $\tilde{k}_2$ are the principal curvatures of the surface $\Gamma(t)$.

We will represent the equation of the surface $\Gamma(t)$ in the form $r = a + \zeta(z, \theta, t)$, assuming that the function ζ and its derivatives are small in absolute magnitude. If $M_0|_\Gamma$ is a point belonging to the surface Γ and $M|_{\Gamma(t)}$ is the point of intersection of the normal to the surface Γ at the point $M_0|_\Gamma$ with the surface $\Gamma(t)$, then we have $\overrightarrow{M_0|_\Gamma M|_{\Gamma(t)}} = -\zeta \mathbf{n}_\Gamma$.

By virtue of the classical result in [2], we obtain

$$(\tilde{k}_1 + \tilde{k}_2) - (k_1 + k_2) = -(k_1^2 + k_2^2)\zeta - \Delta_\Gamma \zeta + O(\zeta^2)$$

where Δ_Γ is the Laplace – Beltrami operator, from which it follows that

$$(\tilde{k}_1 + \tilde{k}_2) = a^{-2} - a^{-2}(\zeta_{\theta\theta} + a^2 \zeta_{zz} + \zeta)$$

apart from terms of the second order of smallness.

On the other hand [2],

$$P(M|_{\Gamma(t)}) - P_0(M_0|_\Gamma) = p(M_0|_\Gamma) - \zeta \operatorname{grad} P_0(M_0|_\Gamma) \cdot \mathbf{n}_\Gamma$$

Laplace's law then ensures that the condition

$$p = \alpha a^{-2}[\zeta_{\theta\theta} + a^2 \zeta_{zz} - (\nu_0 - 1)\zeta] \ \text{ in } \ \Gamma, \ \nu_0 = \rho a^3 \omega_0^2 \alpha^{-1} \tag{1.4}$$

is satisfied. As is well known ζ must be a 2π-periodic function in θ and satisfies the condition

$$\int_\Gamma \zeta d\Gamma = 0 \quad (d\Gamma = a d\theta dz) \tag{1.5}$$

which expresses the constancy of the liquid volume.

Finally we write the condition that the surface $\Gamma(t)$ intersects the container wall at a constant angle. We shall seek the intersection of $\Gamma(t)$ and the wall. Suppose $\zeta = h + \varepsilon(\theta, t)(\zeta = -h + \eta(\theta, t))$ is the neighbourhood of one of the points of intersection, which is in the vicinity of the upper (lower) circle. Then, up to the first order of smallness,

$$\varepsilon = -\operatorname{ctg} \gamma \zeta(h, \theta, t); \quad \eta = \operatorname{ctg} \gamma \zeta(-h, \theta, t)$$

Now, on writing the condition $\mathbf{n}_s \cdot \mathbf{n}'_\Gamma = \cos y$ for $\zeta = \pm h + \varepsilon$, we find

$$\zeta_z = \mp \mu\zeta \quad \text{при} \quad z = \pm h, \quad \mu = f_{zz}(h)\cos^3\gamma/\sin\gamma \tag{1.6}$$

The parameter μ therefore depends on the curvature of the meridian of the container at the points $z = \pm h$. It is positive (negative) if, at these points, the meridian (concavex) with respect to the z axis.

Euler's equation is unchanged when an arbitrary function of time is added to p. This function can be found in such a way that the result will belong to the class

$$\tilde{L}^2(\Gamma) = \left\{\varphi \in L^2(\Gamma): \int_\Gamma \varphi d\Gamma = 0\right\}$$

Condition (1.4) can then be replaced by the condition

$$p = \frac{\alpha}{a^2}[\zeta_{\theta\theta} + a^2\zeta_{zz} - (\nu_0 - 1)\zeta] + \frac{\mu a^2}{4\pi h}\int_0^{2\pi}[\zeta(h,\theta,t) + \zeta(-h,\theta,t)]d\theta \quad \text{на}\,\Gamma \tag{1.7}$$

where $p \in \tilde{L}^2(\Gamma)$.

We introduce the domain $\Omega_0 = \{(\theta, z): 0 < \theta < 2\pi, -h < z < h\}$ and the unbounded operator B_1 which acts on functions from the class

$$\tilde{L}^2(\Omega_0) = \left\{\zeta \in L^2(\Omega_0): \int_{\Omega_0} \zeta d\theta dz = 0\right\}$$

and is defined by the relation

$$B_1\zeta = -\zeta_{\theta\theta} - a^2\zeta_{zz} + (\nu_0 - 1)\zeta - \frac{\mu a^2}{4\pi h}\int_0^{2\pi}[\zeta(h,\theta,t) + \zeta(-h,\theta,t)]d\theta$$

Its domain of definition is such that

$$D(B_1) = \{\zeta \in H^2(\Omega_0); \int_{\Omega_0} \zeta d\theta dz = 0; \ \zeta_z = \mp\mu\zeta \ \text{для}\ z = \pm h;$$

and the traces ζ of orders 0 and 1 coincide at the points $\theta = 0$ and $\theta = 2\pi$ in the sense of the space $L^2(-h, h)$.

We shall seek the bilinear form associated with B_1. For this purpose, we calculate the scalar product $(B_1\zeta, \tilde{\zeta})_{L^2(\Gamma)}$, $\zeta, \tilde{\zeta} \in D(B_1)$. On integrating by parts and taking condition (1.6) into consideration, we obtain the required bilinear form

$$b_1(\zeta,\tilde{\zeta}) = \int_{\Omega_0}[\zeta_\theta\tilde{\zeta}_\theta + a^2\zeta_z\tilde{\zeta}_z + (\nu_0 - 1)\zeta\tilde{\zeta}]d\theta dz + \mu a\int_0^{2\pi}[\zeta(h,\theta)\tilde{\zeta}(h,\theta) + \zeta(-h,\theta)\tilde{\zeta}(-h,\theta)]d\theta \tag{1.8}$$

where $\zeta(\pm h, \theta)$ denotes the traces ζ on the sides $z = \pm h$ of the boundary of the domain Ω_0. The bilinear form $b_1(\zeta_1, \tilde{\zeta})$ is defined for the set of functions

$$V_0 = \{\zeta \in H^1(\Omega_0): \int_{\Omega_0} \zeta d\theta dz = 0;$$

the traces ζ of order 0 coincide at the points $\theta = 0$ and $\theta = 2\pi$ in the sense of the space $L^2(-h, h)\}$.

Equations (1.2)–(1.3) and (1.5)–(1.7) describe small motions of the liquid.

2. THE SUFFICIENT CONDITIONS FOR THE STABILITY OF A RELATIVE EQUILIBRIUM

We multiply Eq. (1.2) scalarly by $\rho\partial w/\partial t$ and integrate with respect Ω. We have

$$\frac{d}{dt}\frac{1}{2}\int_\Omega \rho\left(\frac{\partial w}{\partial t}\right)^2 d\Omega + \int_\Omega \operatorname{grad} p \cdot \frac{\partial w}{\partial t} d\Omega = 0$$

Taking conditions (1.2), (1.3) and (1.7) into consideration, we can write

$$\int_\Omega \operatorname{grad} p \cdot \frac{\partial w}{\partial t} d\Omega = -\int_\Gamma p\zeta_t d\Gamma = \frac{\alpha}{a^2}\int_\Gamma B_1\zeta \cdot \zeta_t d\Gamma = \frac{\alpha}{a^2} b_1(\zeta, \zeta_t) = \frac{\alpha}{a^2}\frac{1}{2}\frac{d}{dt} b_1(\zeta, \zeta)$$

Consequently

$$\frac{d}{dt}\left[\frac{1}{2}\int_\Omega \rho\left(\frac{\partial w}{\partial t}\right)^2 d\Omega + \frac{\alpha}{2a} b_1(\zeta, \zeta)\right] = 0$$

By virtue of the theorem on the change of energy, the last term in the square brackets is the potential of the centrifugal and capillary forces.†

The stability of the relative equilibrium of the liquid with respect to $\|\zeta\|_{L^2(\Omega_0)}$, $\|\zeta_\theta\|_{L^2(\Omega_0)}$, $\|\zeta_z\|_{L^2(\Omega_0)}$, and, hence, with respect to $|\zeta\|_{H^1(\Omega_0)}$ and also with respect to $\|\partial w/\partial t\|_{L^2(\Omega_0)}$ is ensured by the strict coercivity of the bilinear form $b_1(\zeta, \tilde{\zeta})$ (that is, by fact that the quadratic form $b_1(\zeta, \zeta)$ is positive definite (editor's note)) in the set V_0.

Note that stability also holds with respect to $\|\zeta(\pm h, \theta)\|_{L^2(0, 2\pi)}$ when the mapping of the trace of $H^1(\Omega_0)$ into $L^2(0, 2\pi)$ is continuous.

Hence, the stability problem reduces to investigating the coercivity of the bilinear form $b_1(\zeta, \tilde{\zeta})$. It is necessary to distinguish two cases: $\mu \geqslant 0$ and $\mu > 0$.

3. INVESTIGATION OF THE COERCIVITY OF THE FORM $b_1(\zeta, \tilde{\zeta})$. THE CASE WHEN $\mu \geqslant 0$

Suppose

$$\mathcal{T}_1 = \int_{\Omega_0} [\zeta_\theta^2 + a^2\zeta_z^2 + \nu_0\zeta^2] d\theta dz$$

$$\mathcal{T}_2 = \mu a^2 \int_0^{2\pi} [\zeta^2(h, \theta) + \zeta^2(-h, \theta)] d\theta, \quad \mathcal{T}_3 = \int_{\Omega_0} \zeta^2 d\theta dz$$

We denote the function, the lower boundary of which

$$\nu = \inf_{\zeta \in V_0} F(\zeta) \tag{3.1}$$

has to be found, by

$$F(\zeta) = (\mathcal{T}_1 + \mathcal{T}_2)\mathcal{T}_3^{-1}$$

It can be shown that, if $\nu > 1$, the bilinear form $b_1(\zeta, \tilde{\zeta})$ is coercive in V_0. We use the well-known method in [4, 5].

A lower bound of ν exists and it is positive or it vanishes. According to the definition of a lower bound, a sequence $\{\zeta_n\} \in V_0$ exists such that

$$\nu = \lim_{n\to\infty} F(\zeta_n)$$

By the Rellich and Banach–Saks–Mazur theorems [4, 5], it can be proved that the sequence $\{\zeta_n\}$ converges in a weak sense in $H^1(\Omega_0)$ and in a strong sense in $L^2(\Omega_0)$, the limit function U belongs to V_0, the lower bound is attained for $\zeta = U$ and the quantity υ is strictly positive.

We will now find the partial differential equation and the boundary conditions which the quantity U satisfies. According to the definition of ν, we have

$$\mathcal{T}_1 + \mathcal{T}_2 - \nu\mathcal{T}_3 \geqslant 0, \quad \forall \zeta \in V_0$$

†Editor's Note: This potential was called the augmented potential energy by V. V. Rumyantsev. See the publication cited in the footnote on page 605.

We put

$$\zeta = U + \varepsilon\delta\zeta \quad \text{when} \quad \delta\zeta \in V_0, \quad \varepsilon \in R$$

Since $U \in V_0$, variation of this inequality with its subsequent transformation taking the boundary conditions into account gives

$$U_{\theta\theta} - a^2 U_{zz} + (\nu - \nu_0)U + \frac{\mu a^2}{4\pi h}\int_0^{2\pi} [U(h,\theta) + U(-h,\theta)]d\theta = 0 \tag{3.2}$$

Since the coefficients of this elliptic equation are constant, its solutions are functions which belong to the class C^∞.

The boundary conditions are then derived in a classical manner

$$U_z = \mp\mu U, \quad z = \pm h; \quad U_\theta(z, 2\pi) = U_\theta(z, 0) \tag{3.3}$$

which, in conjunction with the integral relation

$$\int_{\Omega_0} U d\theta dz = 0 \tag{3.4}$$

completely determine the solution of the boundary-value problem for which ν is the least eigenvalue.

We will find the eigenvalues of problem (3.2)–(3.4) and the conditions for which the least of the eigenvalues is strictly greater than unity.

4. INVESTIGATION OF THE AUXILIARY EIGENVALUE PROBLEM WHEN $\mu \geqslant 0$

We separate the variables

$$U = \Theta(\theta)Z(z)$$

and put $\Theta', Z' \ldots$ instead of $\Theta_\theta, Z_z, \ldots$. The notation

$$\mathcal{T}_\theta = \int_0^{2\pi} \Theta(\theta)d\theta, \quad \mathcal{T}_z = \int_{-h}^{h} Z(z)dz$$

is used.

The equation and conditions (3.2)–(3.4) then take the form

$$\Theta'' Z + a^2\Theta Z'' + (\nu - \nu_0)\Theta Z + \frac{\mu a^2}{4\pi h}[Z(h) + Z(-h)]\mathcal{T}_\theta = 0 \tag{4.1}$$

$$\Theta(\theta) \equiv \Theta(\theta + 2\pi); \quad Z'(\pm h) = -\mu Z(\mp h); \quad \mathcal{T}_\theta \mathcal{T}_z = 0 \tag{4.2}$$

It is necessary to distinguish between the cases when $\mu > 0$ and $\mu > = 0$.

The case when $\mu > 0$. 1. Suppose $\mathcal{T}_\Theta = 0$. We have

$$\frac{\Theta''}{\Theta} = -\frac{a^2 Z'' + (\nu - \nu_0)Z}{Z} = -n^2, \quad n = 1, 2, \ldots$$

because of periodicity.

Hence, we have

$$\Theta(\theta) = A_n \cos n\theta + B_n \sin n\theta, \quad A_n, \ B_n \text{ – are constants}$$

and the boundary conditions

$$Z'' + \frac{\nu - n^2 - \nu_0}{a^2} Z = 0; \quad Z'(\pm h) = \mp\mu Z(\pm h), \quad n = 1, 2, \ldots$$

This equation can be integrated but, since we are discussing a classical Sturm–Liouville problem, there is no need to do this.

If we put

$$\nu - n^2 - \nu_0 = a^2 \lambda_n^2;$$

then each problem (4.1) has a denumerable set of eigenvalues λ_{nm} such that

$$0 < \lambda_{n1} \leqslant \lambda_{n2} \leqslant \ldots \leqslant \lambda_{nm} \leqslant \ldots, \quad \lambda_{nm} \to +\infty, \quad m \to +\infty$$

The functions Z_{nm} form, for each n, a complete orthogonal system in $L^2(-h, h)$

The eigenvalues, corresponding to problem (3.4), have the form

$$\nu_{nm} = n^2 + \nu_0 + a^2 \lambda_{nm}^2$$

They are all greater than unity.

2. Now suppose $\mathcal{T}_z = 0$. Integrating Eq. (4.1) with respect to θ from 0 to 2π and combining the result with this condition, we obtain Θ = const and the problem

$$Z'' + \frac{\nu - \nu_0}{a^2} Z + \frac{\mu}{2h}[Z(h) + Z(-h)] = 0; \quad Z'(\pm h) = \mp\mu Z(\pm h); \quad \mathcal{T}_z = 0 \tag{4.3}$$

We will now show that (4.3) is a standard eigenvalue problem by demonstrating its variational formulation. On multiplying both sides of the differential equation by the function $\tilde{Z}(z)$, such that

$$\int_{-h}^{h} \tilde{Z}(z) dz = 0$$

integrating the resulting expressions over the interval $[-h, h]$ and then integrating by parts, taking account of the boundary conditions, we obtain the variational formulation in classical form: it is required to find a function $Z \in \tilde{H}^1(-h, h)$ such that

$$\int_{-h}^{h} Z'\tilde{Z}' dz + \mu a^2 [Z(h)\tilde{Z}(h) + Z(-h)\tilde{Z}(-h)] = \frac{\nu - \nu_0}{a^2}(Z, \tilde{Z})_{L^2(-h,h)}$$

$$\forall \tilde{Z} \in \tilde{H}^1(-h, h), \quad \tilde{H}^1(-h, h) = \{Z \in H^1(-h, h) : \mathcal{T}_z = 0\}$$

We introduce the space

$$\tilde{L}^2(-h, h) = \{Z \in L^2(-h, h) : \mathcal{T}_z = 0\}$$

in a natural manner.

The bilinear form

$$a(Z, \tilde{Z}) = \int_{-h}^{h} Z'\tilde{Z}' dz + \mu a^2 [Z(h)\tilde{Z}(h) + Z(-h)\tilde{Z}(-h)]$$

is symmetrical, continuous and coercive in $\tilde{H}^1(-h, h)$ and the injection from $\tilde{H}^1(-h, h)$ into $\tilde{L}^2(-h, h)$ is continuous, dense and compact, as easily follows from the analogous properties of $H^1(-h, h)$ and $L^2(-h, h)$.

Problem (4.3) therefore has a denumerable set of positive eigenvalues ν_{0m} (m =), 1, 2, … which form a non-decreasing sequence that tends to infinity. The eigenvalues of problem (3.4), such that $\nu_{0m} - \nu_0 \geq 0$, correspond to them. The corresponding eigenfunctions Z_{0m} form a complete orthogonal system in $\tilde{L}^2(-h, h)$.

Since 1, $\cos\theta$, $\sin\theta$, …, $\cos n\theta$, $\sin n\theta$, … form a complete orthogonal system in $L^2(0, 2\pi)$, it follows from a classical theorem in [3] that Z_{01}, … , Z_{0m}, … and $Z_{nm}\cos n\theta$, $Z_{nm}\sin n\theta$ form a complete orthogonal system in $L^2(\Omega_0)$. Hence, the method of separation of variables yields all the eigenvalues of problem (3.4) in the case when $\mu > 0$.

The eigenvalues in the case when $\mathcal{T}_\theta = 0$ are greater than unity. We shall seek under what conditions, the eigenvalues when $\mathcal{T}_z = 0$ possess the same property.

We put $a^{-2}(\nu - \nu_0) = \tau^2$ and solve problem (4.3), starting out from

$$Z(z) = A\cos(\tau z) + B\sin(\tau z) - \frac{\mu}{2h\tau^2}[Z(h) + Z(-h)]$$

Assuming $\tau h = \varkappa$, we obtain the equations for the eigenvalues

$$\operatorname{tg}\varkappa = -\frac{\varkappa}{\mu h}; \quad \frac{1}{\varkappa} - \operatorname{ctg}\varkappa = -\frac{\varkappa}{\mu h}$$

Graphical analysis shows that the least value $\varkappa_1$ of the unknown $\varkappa$ is located between $\pi/2$ and π. Assuming $\tau_1 = \varkappa_1/h$, we find the least eigenvalue in the case when $\mathcal{T}_z = 0$

$$\nu_1 = \nu_0 + a^2\tau_1^2$$

We conclude that when $\mu > 0$, if

$$\nu_0 + a^2\tau_1^2 > 1 \tag{4.4}$$

then all of the eigenvalues of problem (3.4) are greater than unity.

The case when $\mu = 0$. The arguments are analogous in this case but somewhat simpler. When $\mathcal{T}_\theta = 0$, the eigenvalues ν are such that

$$\nu \geqslant n^2 + \nu_0 > 1$$

When $\mathcal{T}_z = 0$, simple calculations show that, when the condition

$$\nu_1' = \nu_0 + \left(\frac{\pi a}{2h}\right)^2 > 1 \tag{4.5}$$

is satisfied, the least, and together with it, all of the remaining eigenvalues are greater than unity.

The sufficient condition for the stability of a relative equilibrium. We will now show that, in the case when $\mu \geqslant 0$, if the lower bound ν of the ratio from relation (3.1) is greater than unity, the bilinear form $b_1(\zeta, \bar{\zeta})$ is coercive in V_0, and it follows from this that the relative equilibrium position of the liquid is stable.

From the definition of ν, we have

$$\mathcal{T}_1 + \mathcal{T}_2 \geqslant \nu\mathcal{T}_3 \quad \forall\zeta \in V_0$$

Using this inequality, we conclude that, if ε is a number such that $0 < \varepsilon < 1$, then

$$b_1(\zeta, \zeta) \geqslant \varepsilon\{\mathcal{T}_1 + \mathcal{T}_2\} + [(1-\varepsilon)\nu - 1]\mathcal{T}_3$$

Then, on choosing $0 < \varepsilon < 1 - \upsilon^{-1}$, which is permissible, we have

$$b_1(\zeta, \zeta) \geqslant \varepsilon[\mathcal{T}_1 + \mathcal{T}_2], \quad \forall\zeta \in V_0$$

whence it follows that the form $b_1(\zeta, \bar{\zeta})$ is coercive in V_0. Hence, when $\mu > 0$ ($\mu = 0$, respectively), the inequality (4.4) ((4.5), respectively) is a sufficient condition for the stability of the relative equilibrium of the fluid.

Remark. If the container is fixed ($\omega_0 = 0$), it is possible to repeat the arguments of the preceding discussion, since the norm

$$\left[\int_{\Omega_0}(\zeta_\theta^2+a^2\zeta_z^2)d\theta dz\right]^{1/2}$$

in V_0 is equivalent to the norm in $H^1(\Omega_0)$.

When $\mu > 0$ and $\omega_0 = 0$, the condition $a\tau_1 > 1$ is also a sufficient condition for the stability of the equilibrium. Hence, uniform rotation stabilizes the motion.

When $\mu = 0$ and $\omega_0 = 0$, there is an eigenvalue $\nu = 1$ in the case when $\mathcal{T}_\theta = 0$ and it is impossible to draw any conclusion regarding the stability of the equilibrium.

5. INVESTIGATION OF THE EIGENVALUE PROBLEM WHEN $\mu < 0$

Reduction to an eigenvalue problem.

Suppose

$$G(\zeta)=\mathcal{T}_1(\mathcal{T}_3-\mathcal{T}_2)^{-1}$$

we shall now seek

$$\nu=\inf_{\zeta\in V_0}G(\zeta) \tag{5.1}$$

A lower bound exists, it is positive and equal to zero. According to the definition of a lower bound, a sequence $\{\zeta_n \in V_0\}$ exists such that

$$\nu=\lim_{n\to\infty}G(\zeta_n)$$

As in the case when $\mu \geqslant 0$, it can be shown that the sequence $\{\zeta_n\}$ converges in a weak sense in $H^1(\Omega_0)$ and in a strong sense in $L^2(\Omega_0)$, the limit function U belongs to V_0, the lower bound is attained for $\theta = U$, the inequality $\nu > 0$ is satisfied and the function U satisfies the partial differential equation and the conditions

$$u_{\theta\theta}+a^2u_{zz}+(\nu-\nu_0)u+\frac{\nu\mu a^2}{4\pi h}\int_0^{2\pi}[u(h,\theta)+u(-h,\theta)]d\theta=0 \tag{5.2}$$

$$u(z,0)=u(z,2\pi),\quad u_\theta(z,0)=u_\theta(z,2\pi);\quad u_z=\mp\nu\mu u\quad \text{for}\quad z=\pm h$$

$$\int_{\Omega_0}ud\theta dz=0$$

where ν is the least eigenvalue of the problem.

Problem (5.2) is a Steklov problem since the eigenvalues ν occur in the boundary conditions.

We will now show that we are dealing here with a standard eigenvalue problem On multiplying the equation from (5.2) by $v \in V_0$, integrating with respect to Ω_0 and taking the periodicity and the boundary conditions into account, we obtain the variational formulation of the problem: it is required to find a function $u \in V_0$ and a real number ν such that

$$\int_{\Omega_0}[u_\theta v_\theta+a^2u_zv_z+\nu_0uv]d\theta dz=$$

$$=\left[\int_{\Omega_0}uvd\theta dz-\mu a^2\int_0^{2\pi}[u(h,\theta)v(h,\theta)+u(-h,\theta)v(-h,\theta)]d\theta\right]\nu,\quad \forall\in V_0 \tag{5.3}$$

The coefficient of ν is a scalar product since $\mu < 0$.

We therefore introduce the space $\mathcal{H}$ which competes V_0 with respect to the norm associated with the scalar product

$$(u,v)_{\mathcal{H}}=\int_{\Omega_0}uvd\theta dz-\mu d^2\int_0^{2\pi}[u(h,\theta)v(h,\theta)+u(-h,\theta)v(-h,\theta)]d\theta$$

The first term in (5.3) is a bilinear symmetric, continuous and coercive form in V_0. The injection of V_0 into $\mathcal{H}$ is dense and continuous. We will now show that it is compact. For this purpose, we consider a sequence $\{u_n\} \in V_0$ which converges weakly to u in V_0. The sequence $\{u_n\}$ converges strongly to u in $L^2(\Omega_0)$ and the sequence $\{u_n(\pm h, \theta)\}$ converges strongly to $u(\pm h, \theta)$ in $L^2(0, 2\pi)$ by virtue of the Sobolev–Kondrashov theorem. Hence, the sequence $\{u_n\}$ converges strongly to u in the space $\mathcal{H}$.

Problem (5.2) admits of a denumerable set of positive eigenvalues which form a non-decreasing sequence that tends to infinity. The corresponding eigenfunctions form a complete orthogonal system in the space $\mathcal{H}$.

Investigation of the auxiliary eigenvalue problem. On seeking solutions of problem (5.2) in the form $u = \Theta(\theta)Z(z)$, we obtain

$$\Theta''Z + a^2\Theta Z'' + (\nu - \nu_0)\Theta Z + \frac{\nu\mu a^2}{4\pi h}[Z(h) + Z(-h)]\mathcal{T}_\theta = 0$$
$$\Theta(\theta) \equiv \Theta(\theta + \pi), \quad Z'(\pm h) = -\nu\mu Z(\pm h), \quad \mathcal{T}_\theta \mathcal{T}_z = 0 \tag{5.4}$$

The case when $\mathcal{T}_\theta = 0$. We have

$$\Theta(\theta) = A_n \cos n\theta + B_n \sin n\theta$$

and the equations

$$Z'' + (\nu - n^2 - \nu_0)a^{-2}Z = 0, \quad Z'(\pm h) = -\nu\mu Z(\pm h) \quad n = 1, 2, \ldots \tag{5.5}$$

As above, it is clear that the Steklov problems are standard eigenvalue problems. For each n, the eigenvalues $\nu_{nm} \to +\infty$ as $m \to \infty$. We now find the least among them which are strictly less than $n^2 + \nu_0$.

We put

$$\nu - n^2 - \nu_0 = -a^2\lambda_n^2 < 0$$

and slove problem (5.5). We obtain the equations

$$\text{th}(\lambda_n h) = -\nu\mu/\lambda_n, \quad \text{cth}(\lambda_n h) = -\nu\mu/\lambda_n$$

Simple graphical analysis gives the following results. The first equation has a root λ_n^0. The second equation does not have a root if the quantity $-\mu h(n^2 + \nu_0)$ is less than unity and has a single root if this quantity is greater than unity. We conclude from this that the least eigenvalue is equal to

$$\nu_{n1} = n^2 + \nu_0 - a^2\lambda_n^{0^2}$$

This quantity is greater than unity if

$$\lambda_n^0 < \varkappa = a^{-1}\sqrt{n^2 - 1 + \nu_0}$$

This condition is equivalent to the inequality th $\varkappa > -\mu h/\varkappa$ so that $\varkappa > \varkappa_0(\mu h)$, where the quantity $\varkappa_0(\mu h)$, which is independent of n, is the abscissa of the point of intersection of the curves $y = \text{th}(\varkappa)$, $y = -\mu h/\varkappa$. Hence, the condition $\nu_{n1} > 1$ can be written in the form

$$n^2 - 1 + \nu_0^2 > a^2h^{-2}\varkappa_0^2(\mu h)$$

Finally, all the eigenvalues ν_{nm} $(n, m = 1, 2, 3\ldots)$ are greater than unity if the inequality

$$\nu_0 > a^2h^{-2}\varkappa_0^2(\mu h) \tag{5.6}$$

is satisfied.

The case when $\mathcal{T}_z = 0$. In this case, we obtain that the quantity Θ is constant, and the problem has the form

$$Z'' + \frac{1}{a^2}(\nu - \nu_0)Z + \frac{\nu\mu}{2h}[Z(h) + Z(-h)] = 0, \quad Z'(\pm h) = \mp\nu\mu Z(\pm h); \quad \mathcal{T}_z = 0 \tag{5.7}$$

We will now show that this Steklov problem is a standard eigenvalue problem and, by solving problem (5.7), we shall investigate the least of its eigenvalues.

If a least eigenvalue of the initial problem exists which is less than ν_0, then it is simultaneously an eigenvalue of problem (5.7).

On putting $\nu - \nu_0 = -a^2\lambda^2 < 0$ and solving problem (5.7), we obtain

$$\operatorname{th} \varkappa = -\varkappa(\mu h\nu)^{-1}, \quad \operatorname{cth} \varkappa - \varkappa^{-1} = -\varkappa(\mu h\nu)^{-1} \quad (\varkappa = \lambda h) \tag{5.8}$$

Simple graphical analysis shows that, if $\nu_0 \leqslant -(\mu h)^{-1}$, these equations do not have solutions and, consequently, eigenvalues which are less than ν_0 do not exist. If $\nu > -(\mu h)^{-1}$, the least eigenvalue is equal to

$$\nu_0' = \nu_0 - a^2 h^{-2} \varkappa_0'^2$$

where $\varkappa_0'$ is the root of the first of the equations (5.8). The condition $\upsilon_0' > 1$ is only satisfied when

$$\varkappa_0' < h a^{-1}\sqrt{\nu_0 - 1}$$

or

$$\operatorname{th} u < -u(\mu h)^{-1} \quad (u = h a^{-1}\sqrt{\nu_0 - 1})$$

Moreover, $\nu_0' > 1$ if the conditions

$$-(\mu h)^{-1} \geqslant 1, \quad \nu_0 > -(\mu h)^{-1}$$

or

$$-(\mu h)^{-1} < 1, \quad \nu_0 > 1 + a^2 h^{-2} \delta^2(\mu h)$$

are satisfied, where $\delta(\mu h)$ is the root of the equation $\operatorname{th} u = -(\mu h)^{-1} u$.

On combining these conditions with condition (5.6), we conclude that: if the conditions

$$-(\mu h)^{-1} \geqslant 1, \quad \nu_0 > \max(-(\mu h)^{-1}, \quad a^2 h^{-2} \varkappa_0^2(\mu h)) \tag{5.9}$$

or the conditions

$$-(\mu h)^{-1} < 1, \quad \nu_0 > \max(1 + a^2 h^{-2} \delta^2(\mu h), \ a^2 h^{-2} \varkappa_0^2(\mu h)) \tag{5.10}$$

are satisfied, then all of the eigenvalues of problem (3.4) are greater than unity.

We will now investigate the case when $\nu_0 \leqslant -(\mu h)^{-1}$. In this case, the least eigenvalue of problem (5.7) is greater than or equal to υ_0. On putting $\nu - \nu_0 = 0$ in (5.7), we find, in an elementary manner, the two values of $\nu = -(\mu h)^{-1}$, $\nu = -3(\mu h)^{-1}$. We then obtain: if

$$-(\mu h)^{-1} > 1, \quad a^2 h^{-2} < -(\mu h)^{-1} \varkappa_0^{-2}(\mu h) \tag{5.11}$$

and if

$$\nu_0 = -(\mu h)^{-1} \tag{5.12}$$

then all of the eigenvalues of problem (3.4) are greater than unity.

Finally, we investigate the case when $\nu - \nu_0 > 0$. On putting $\nu - \nu_0 = a^2\lambda^2 > 0$ and solving system (5.7), we obtain the equations

$$\operatorname{tg} \varkappa = -\varkappa(\mu h\nu)^{-1}, \quad \varkappa^{-1} - \operatorname{ctg} \varkappa = -\varkappa(\mu h\nu)^{-1}$$

Graphical analysis shows that these equations have a denumerable set of different roots, the least of which ξ_0 is the least root of the first equation The least eigenvalue of problem (5.7)

$$\tilde{\nu}_0 = \nu_0 + a^2h^{-2}\xi_0^2$$

corresponds to it. The condition $\tilde{\nu}_0 > 1$ is written as

$$a^2h^{-2}\xi_0^2 > 1 - \nu_0$$

This condition is satisfied if $\nu_0 \geqslant 1$. If $\nu_0^2 < 1$, then it can be written as

$$\xi_0 > ha^{-1}\sqrt{1-\nu_0}$$

Proceeding as above, we see that $\tilde{\nu}_0 > 1$ if

$$-(\mu h)^{-1} > 1 \text{ and } 1 - a^2h^{-2}\beta^2(\mu, h) < \nu_0 < 1$$

where $\beta(\mu h)$ is the root of the equation $\operatorname{tg} u = -u(\mu h)^{-1}$, which is located between 0 and $\pi/2$.

On combining these inequalities with condition (5.6), we have the following result: if the conditions

$$\begin{gathered} -(\mu h)^{-1} > 1, \quad a^2h^{-2} < -(\mu h)^{-1}\varkappa_0^2(\mu h) \\ \max(1 - a^2h^{-2}\beta^2(\mu h),\ a^2h^{-2}\varkappa_0^2(\mu h)) < \nu_0 < -(\mu h)^{-1} \end{gathered} \tag{5.13}$$

are satisfied, all the eigenvalues of problem (3.4) are greater than unity.

The sufficient conditions for the stability of a relative equilibrium. We will now prove that when $\mu < 0$, if the lower bound υ of the ratio in relation (5.1) is greater than unity, the bilinear form is coercive in V_0 and the position of relative equilibrium of the liquid is stable.

From the definition of ν, we have

$$\mathcal{T}_1 \geqslant \nu[\mathcal{T}_3 - \mathcal{T}_2], \quad \forall\zeta \in V_0$$

By virtue of this inequality in the case of a number ε which is such that $0 < \varepsilon < 1$, the relation

$$b_1(\zeta, \zeta) = \varepsilon\mathcal{T}_1 + [(1-\varepsilon)\nu - 1]\{\mathcal{T}_3 - \mathcal{T}_2\}$$

is satisfied. On choosing ε such that $0 < \varepsilon < 1 - \nu^{-1}$, we have

$$b_1(\zeta, \zeta) \geqslant \varepsilon\mathcal{T}_1, \quad \forall\zeta \in V_0$$

which implies the coercivity of the form $b_1(\zeta, \tilde{\zeta})$ in V_0.

Finally, if $\mu < 0$ and either one of conditions (5.9) and (5.10) or conditions (5.11) and (5.12) or conditions (5.13) are satisfied, the relative equilibrium position of the liquid is stable.

Hence, the value of the parameter μ and, together with it, also the curvatures of the meridian at the point of contact with the liquid surface, have a substantial effect on the nature of the stability of the relative equilibrium. Here, the shapes of containers for which $-(\mu h)^{-1} > 1, -(\mu h)^{-1} < 1$ when $\mu < 0$ can be indicated.

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Translated by E.L.S.